

# A new third-order cosmic shear statistic: separating E-/B-mode correlations on a finite interval

Elisabeth Krause,<sup>1★</sup> Peter Schneider<sup>2</sup> and Tim Eifler<sup>3</sup>

<sup>1</sup>*Department of Astrophysics, California Institute of Technology, MC 249-17, Pasadena, CA 91125, USA*

<sup>2</sup>*Argelander Institut für Astrophysik, Universität Bonn, Auf dem Hügel 71, 53121 Bonn, Germany*

<sup>3</sup>*Center for Cosmology and Astro-Particle Physics, The Ohio State University, 191 West Woodruff Avenue, Columbus, OH 43210, USA*

Accepted 2012 March 26. Received 2012 March 25; in original form 2012 January 23

## ABSTRACT

Decomposing the shear signal into E and B modes properly, i.e. without leakage of B modes into the E-mode signal and vice versa, has been a long-standing problem in weak gravitational lensing. At the two-point level this problem was resolved by developing the so-called ring statistic, and later the COSEBIs; however, extending these concepts to the three-point level is far from trivial. Currently used methods to decompose shear three-point correlation functions (3PCFs) into E and B modes require knowledge of the 3PCF down to arbitrary small scales. This implies that the 3PCF needs to be modelled on scales smaller than the minimum separation of two galaxies and will subsequently be biased towards the model, or, in the absence of a model, the statistic is affected by E-/B-mode leakage (or mixing). In this paper, we derive a new third-order E-/B-mode statistic that performs the decomposition using the 3PCF only on a finite interval, and thereby is free of any E-/B-mode leakage while at the same time relying solely on information from the data. In addition, we relate this third-order ring statistic to the convergence field, thereby enabling a fast and convenient calculation of this statistic from numerical simulations. We note that our new statistic should be applicable to corresponding E-/B-mode separation problems in the cosmic microwave background polarization field.

**Key words:** cosmology: theory – large-scale structure of Universe.

## 1 INTRODUCTION

Cosmic shear, the distortion of light from distant galaxies by the tidal gravitational field of the intervening large-scale structure, is an excellent tool to probe the matter distribution in the universe. The statistics of the image distortions are related to the statistical properties of the large-scale matter distribution and the geometry of the universe, and can thereby be used to constrain cosmology. Current results already demonstrate the power of cosmic shear observations at constraining the clustering amplitude  $\sigma_8$  and the matter density  $\Omega_m$  (e.g. Fu et al. 2008; Schrabback et al. 2010; Huff et al. 2011). Furthermore, cosmic shear provides an ideal tool to study dark energy through measuring the growth of structure with large future surveys such as KIDS,<sup>1</sup> DES,<sup>2</sup> LSST<sup>3</sup> (Ivezic et al. 2008) or *Euclid*<sup>4</sup> (Laureijs et al. 2011). The large volume probed by these surveys will enable us to measure not only the power spectrum, but also higher order statistics with unprecedented precision. As the evolved density field is non-Gaussian, the three-point correlation function (3PCF) and its Fourier space equivalent, the bispectrum, contain significant cosmological information complementary to the more commonly used two-point statistics and are a powerful tool for breaking parameter degeneracies (Takada & Jain 2004).

The upcoming weak lensing experiments will limit the statistical uncertainties to the per cent level. In order to extract cosmological information from these cosmic shear experiments, the increased data quality needs to be accompanied by a thorough treatment of a wide range of systematic errors, from photometric redshifts and galaxy shape measurements to the removal of astrophysical contaminants.

★E-mail: ekrause@tapir.caltech.edu

<sup>1</sup> <http://www.astro-wise.org/projects/KIDS>

<sup>2</sup> <http://www.darkenergysurvey.org/>

<sup>3</sup> <http://www.lsst.org/lsst/>

<sup>4</sup> <http://sci.esa.int/euclid>

If the shear estimated from observed galaxy shapes is solely caused by gravitational lensing, then it should consist only of a ‘gradient component’, the so-called E-mode shear. B modes (or curl components) cannot be generated by gravitational light deflection to leading order, and higher order corrections are expected to be very small. Hence observing any B-mode pattern indicates remaining systematics in the shear analysis.

Decomposing the observed shear field directly into E/B modes (e.g. Bunn et al. 2003) is complicated by the complex mask geometry of weak lensing observations. At the two-point statistic level, an E/B-mode decomposition is commonly performed using the aperture mass dispersion (Schneider et al. 1998) and related measures (e.g. Crittenden et al. 2002), which can be calculated from the measured shear two-point correlation function (2PCF) and is thus not affected by the masking geometry. However, these methods assume that the 2PCF is known either from  $\theta = 0$  to some finite angular value (aperture mass dispersion) or to arbitrarily large separations. However, in reality the 2PCF can only be measured on a finite interval  $[\theta_{\min}, \theta_{\max}]$ , where the lower boundary is caused by inability to measure the shape of image pairs with very small angular separation. As Kilbinger, Schneider & Eifler (2006) pointed out, lack of shear-correlation measurements on small scales leads to an underestimation of the aperture mass dispersion on small scales and causes an apparent mixing of E and B modes with this type of estimator. Schneider & Kilbinger (2007), Eifler, Schneider & Krause (2010) and Schneider, Eifler & Krause (2010) develop statistical measures for an exact E/B-mode decomposition based on 2PCFs known only on a finite interval  $[\theta_{\min}, \theta_{\max}]$ .

At the three-point statistic level, Jarvis, Bernstein & Jain (2004) and Schneider, Kilbinger & Lombardi (2005) introduced E/B-mode separating shear measures which assume knowledge of the 3PCF down to arbitrarily small scales. Shi, Schneider & Joachimi (2011) derived a general condition for the E/B-mode decomposition of lensing three-point statistics, but the construction of filter functions with finite support based on this condition is far from being straightforward. In this paper, we derive an extension of the 2PCF ring statistic (Schneider & Kilbinger 2007; Eifler et al. 2010) to an exact E/B-mode decomposition of shear 3PCFs on a finite interval.

In order to constrain cosmology with third-order shear statistics, it is important to obtain the corresponding predictions from a large suite of cosmological numerical simulations in a reasonable time and with limited computational effort. We facilitate this by giving an expression of the third-order ring statistic in terms of the convergence field, thereby avoiding the time-consuming calculation of the shear 3PCF for each simulation.

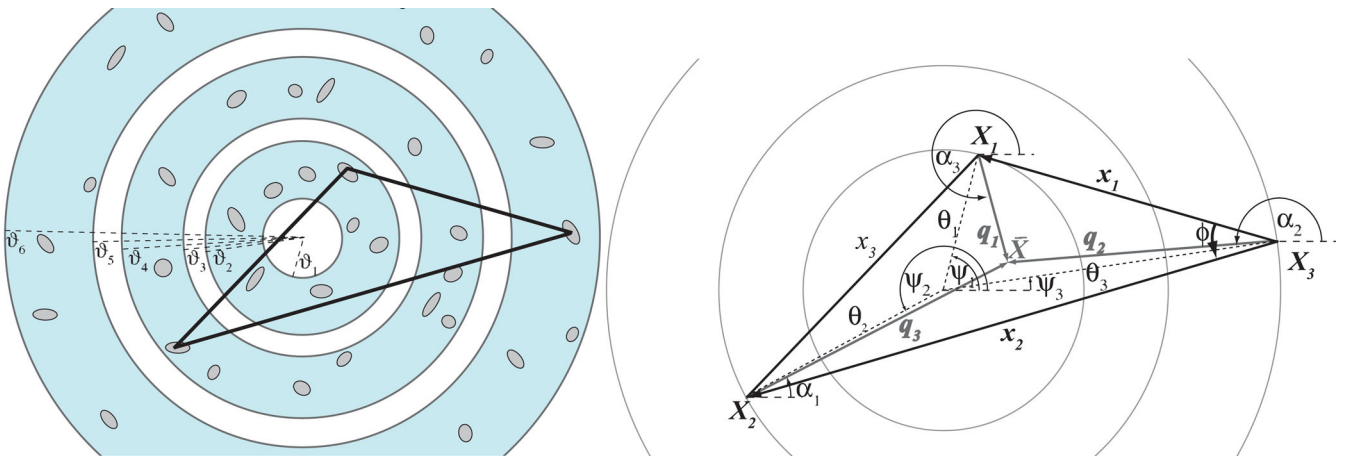
## 2 SHEAR THREE-POINT CORRELATION FUNCTIONS

We first introduce the shear 3PCF: consider a triangle in the complex plane with vertices  $X_i$  and let  $\gamma_\mu(X_i)$ ,  $\mu = 1, 2$ , be the Cartesian components of the shear at point  $X_i$ . Unless otherwise noted, we will assume that the triangle is oriented such that  $X_1, X_2, X_3$  are ordered counterclockwise around the triangle. We define  $\mathbf{x}_1 = X_1 - X_3$  and  $\mathbf{x}_2 = X_2 - X_3$  to be the sides of this triangle (cf. Fig. 1). We will use  $x_i$  to refer to complex numbers or vectors interchangeably, and denote their magnitude as  $x_i$ .

The Cartesian components of the shear 3PCF are defined as

$$\gamma_{\mu\nu\lambda}(\mathbf{x}_1, \mathbf{x}_2) \equiv \langle \gamma_\mu(X_1) \gamma_\nu(X_2) \gamma_\lambda(X_3) \rangle, \quad (1)$$

where we have assumed that the shear field is statistically homogeneous so that  $\gamma_{\mu\nu\lambda}$  depends only on the side vectors  $\mathbf{x}_i$ . Since one cannot form a trilinear scalar from the product of three shears, the behaviour of the Cartesian components of the shear 3PCF under rotations is complicated. In order to write the 3PCF in terms of tangential ( $\gamma_t$ ) and cross components ( $\gamma_\times$ ) of the shear which are parity eigenstates



**Figure 1.** Left: concept of the third-order ring statistic  $\langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle(\vartheta_1, \vartheta_2, \dots, \vartheta_6)$ . E/B modes are separated on a finite interval by correlating the shear of galaxy triplets located within three concentric annuli, as illustrated by the thick black triangle. The minimum separation between galaxies in the above geometry is  $\min(\vartheta_3 - \vartheta_2, \vartheta_5 - \vartheta_4)$  and the maximum separation is  $\vartheta_6 + \vartheta_4$ . Right: geometry of a triangle in the third-order ring statistic. The triangle vertices  $X_j$  are located on three concentric circles of radius  $\theta_j$  and have polar angles  $\psi_j$ . The triangle centroid is  $\bar{X}$ . In the centroid projection the shear at each  $X_j$  is projected on to the centroid along the direction  $\mathbf{q}_j$ , the line connecting  $X_j$  with the centroid.  $\varphi_j$  is the orientation angle of vector  $\mathbf{q}_j$ . Finally,  $\phi$  is the inner angle of the triangle at  $X_3$  which we will use when parametrizing a triangle in terms of two side lengths  $x_1, x_2$  and angle  $\phi$ .

and have relatively simple transformation properties, one can project the complex Cartesian shear  $\gamma^c = \gamma_1 + i\gamma_2$  into tangential and cross components with respect to a chosen direction  $\mathbf{a}_i$  with polar angle  $\alpha_i$ :

$$\gamma(\mathbf{X}_i; \alpha_i) \equiv \gamma_t(\mathbf{X}_i; \alpha_i) + i\gamma_\times(\mathbf{X}_i; \alpha_i) = -[\gamma_1(\mathbf{X}_i) + i\gamma_2(\mathbf{X}_i)] e^{-2i\alpha_i} = -\gamma^c(\mathbf{X}_i) e^{-2i\alpha_i} = -\gamma^c(\mathbf{X}_i) a_i^{*2}/a_i^2. \quad (2)$$

If the directions of projection  $\alpha_i$  are defined in terms of the vertices  $\mathbf{X}_i$  and thus do not depend on an external coordinate system, then the tangential and cross shear are invariant under rotations of the triangle (Schneider & Lombardi 2003; Takada & Jain 2003; Zaldarriaga & Scoccimarro 2003), and the 3PCF of these shear projections will only depend on the side *lengths*  $x_i$  and the orientation of the triangle (clockwise or counterclockwise). In the following, we will use the centroid projection, where the shear at vertex  $\mathbf{X}_i$  is projected along the direction  $\mathbf{q}_i$  connecting  $\mathbf{X}_i$  with the centroid  $\bar{\mathbf{X}} = (\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3)/3$ , and  $\alpha_i$  is the polar angle of this projection direction (see Fig. 1 for an illustration).

Following Schneider & Lombardi (2003), we define the (complex) natural components of the 3PCF which have relatively simple transformation properties:

$$\begin{aligned} \Gamma^{(0)}(x_1, x_2, x_3) &\equiv \langle \gamma(\mathbf{X}_1; \alpha_1) \gamma(\mathbf{X}_2; \alpha_2) \gamma(\mathbf{X}_3; \alpha_3) \rangle = -\langle \gamma^c(\mathbf{X}_1) \gamma^c(\mathbf{X}_2) \gamma^c(\mathbf{X}_3) \rangle e^{-2i(\alpha_1 + \alpha_2 + \alpha_3)}, \\ \Gamma^{(1)}(x_1, x_2, x_3) &\equiv \langle \gamma^*(\mathbf{X}_1; \alpha_1) \gamma(\mathbf{X}_2; \alpha_2) \gamma(\mathbf{X}_3; \alpha_3) \rangle = -\langle \gamma^{c*}(\mathbf{X}_1) \gamma^c(\mathbf{X}_2) \gamma^c(\mathbf{X}_3) \rangle e^{-2i(-\alpha_1 + \alpha_2 + \alpha_3)}, \\ \Gamma^{(2)}(x_1, x_2, x_3) &\equiv \langle \gamma(\mathbf{X}_1; \alpha_1) \gamma^*(\mathbf{X}_2; \alpha_2) \gamma(\mathbf{X}_3; \alpha_3) \rangle, \quad \Gamma^{(3)}(x_1, x_2, x_3) \equiv \langle \gamma(\mathbf{X}_1; \alpha_1) \gamma(\mathbf{X}_2; \alpha_2) \gamma^*(\mathbf{X}_3; \alpha_3) \rangle. \end{aligned} \quad (3)$$

$\Gamma^{(0)}$  is invariant under cyclic permutations of arguments; the other three components transform into each other:  $\Gamma^{(1)}(x_1, x_2, x_3) = \Gamma^{(2)}(x_3, x_1, x_2) = \Gamma^{(3)}(x_2, x_3, x_1)$ , etc. A different parametrization of oriented triangles is in terms of two sides and their inner angle, e.g.  $x_1, x_2$  and  $\phi$  (cf. Fig. 1). We choose the convention  $\phi \in [-\pi, \pi]$ , such that  $\phi > 0$  corresponds to  $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$  being ordered counterclockwise ('positive orientation') and  $\phi < 0$  corresponds to clockwise ordering ('negative orientation').

### 3 E-/B-MODE SEPARATION

To construct integrals which separate third-order E- and B-mode correlations, we start from the circle statistic  $\mathcal{C}(\theta)$  (Crittenden et al. 2002; Schneider, van Waerbeke & Mellier 2002) which geometrically separates E and B modes by measuring the mean tangential and cross components of the shear on a circle of radius  $\theta$  around the origin

$$\mathcal{C}(\theta) = \mathcal{C}_t(\theta) + i\mathcal{C}_\times(\theta) = \frac{1}{2\pi} \int_0^{2\pi} d\psi (\gamma_t + i\gamma_\times)(\theta, \psi; \psi) = -\frac{1}{2\pi} \int_0^{2\pi} d\psi e^{-2i\psi} \gamma^c(\theta, \psi), \quad (4)$$

where  $\psi$  is the polar angle on the circle, and in the last step we have rotated the tangential and radial shear into the Cartesian components. Following Schneider & Kilbinger (2007), we now consider the shear inside an annulus  $\vartheta_1 \leq \theta \leq \vartheta_2$  and define the ring statistic  $\mathcal{R}$ ,

$$\mathcal{R} = \mathcal{R}_t + i\mathcal{R}_\times = \int_{\vartheta_1}^{\vartheta_2} d\theta W(\theta; \vartheta_1, \vartheta_2) \mathcal{C}(\theta), \quad (5)$$

which is a function of two radii  $\vartheta_i$  and  $\vartheta_j$ , and where  $W(\theta; \vartheta_1, \vartheta_2)$  is a normalized weight function

$$\int_{\vartheta_1}^{\vartheta_2} d\theta W(\theta; \vartheta_1, \vartheta_2) = 1, \quad (6)$$

and  $W = 0$  outside the annulus, i.e. if  $\theta < \vartheta_1$  or  $\theta > \vartheta_2$ . From this definition, we construct the third-order ring statistic as the correlation of the weighted mean shear in three concentric annuli with radii  $\vartheta_1 \leq \theta_1 \leq \vartheta_2 < \vartheta_3 \leq \theta_2 \leq \vartheta_4 < \vartheta_5 \leq \theta_3 \leq \vartheta_6$  (cf. Fig. 1),

$$\langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle(\boldsymbol{\vartheta}) = \int_{\vartheta_1}^{\vartheta_2} d\theta_1 W(\theta_1; \vartheta_1, \vartheta_2) \int_{\vartheta_3}^{\vartheta_4} d\theta_2 W(\theta_2; \vartheta_3, \vartheta_4) \int_{\vartheta_5}^{\vartheta_6} d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) \langle \mathcal{C}(\theta_1) \mathcal{C}(\theta_2) \mathcal{C}(\theta_3) \rangle, \quad (7)$$

$$\langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) = \int_{\vartheta_1}^{\vartheta_2} d\theta_1 W(\theta_1; \vartheta_1, \vartheta_2) \int_{\vartheta_3}^{\vartheta_4} d\theta_2 W(\theta_2; \vartheta_3, \vartheta_4) \int_{\vartheta_5}^{\vartheta_6} d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) \langle \mathcal{C}^*(\theta_1) \mathcal{C}(\theta_2) \mathcal{C}(\theta_3) \rangle, \quad (8)$$

where we have used  $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_6)$  to denote a six-tuple of radii. Expanding these correlators in terms of the mean tangential and cross shear yields

$$\langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle(\boldsymbol{\vartheta}) = [\langle \mathcal{R}_t \mathcal{R}_t \mathcal{R}_t \rangle - \langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_t \rangle - \langle \mathcal{R}_\times \mathcal{R}_t \mathcal{R}_\times \rangle - \langle \mathcal{R}_t \mathcal{R}_\times \mathcal{R}_\times \rangle + i(-\langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_\times \rangle + \langle \mathcal{R}_\times \mathcal{R}_t \mathcal{R}_t \rangle + \langle \mathcal{R}_t \mathcal{R}_\times \mathcal{R}_t \rangle + \langle \mathcal{R}_t \mathcal{R}_t \mathcal{R}_\times \rangle)](\boldsymbol{\vartheta}), \quad (9)$$

$$\langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) = [\langle \mathcal{R}_t \mathcal{R}_t \mathcal{R}_t \rangle + \langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_t \rangle + \langle \mathcal{R}_\times \mathcal{R}_t \mathcal{R}_\times \rangle - \langle \mathcal{R}_t \mathcal{R}_\times \mathcal{R}_\times \rangle + i(\langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_\times \rangle - \langle \mathcal{R}_\times \mathcal{R}_t \mathcal{R}_t \rangle + \langle \mathcal{R}_t \mathcal{R}_\times \mathcal{R}_t \rangle + \langle \mathcal{R}_t \mathcal{R}_t \mathcal{R}_\times \rangle)](\boldsymbol{\vartheta}). \quad (10)$$

Note that the imaginary parts of equations (9) and (10) vanish in the absence of parity-violating modes.

We analogously define the correlators  $\langle \mathcal{R}\mathcal{R}^* \mathcal{R} \rangle$  and  $\langle \mathcal{R}\mathcal{R}\mathcal{R}^* \rangle$  and separate E and B modes via

$$\langle \mathcal{R}_E^3 \rangle(\boldsymbol{\vartheta}) \equiv \langle \mathcal{R}_t \mathcal{R}_t \mathcal{R}_t \rangle(\boldsymbol{\vartheta}) = \frac{1}{4} \text{Re} [\langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle + \langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R}\mathcal{R}^* \mathcal{R} \rangle + \langle \mathcal{R}\mathcal{R}\mathcal{R}^* \rangle](\boldsymbol{\vartheta}), \quad (11)$$

$$\langle \mathcal{R}_B^3 \rangle(\boldsymbol{\vartheta}) \equiv \frac{1}{3} [\langle \mathcal{R}_t \mathcal{R}_\times \mathcal{R}_\times \rangle + \langle \mathcal{R}_\times \mathcal{R}_t \mathcal{R}_\times \rangle + \langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_t \rangle](\boldsymbol{\vartheta}) = \frac{1}{4} \text{Re} [-3\langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle + \langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R}\mathcal{R}^* \mathcal{R} \rangle + \langle \mathcal{R}\mathcal{R}\mathcal{R}^* \rangle](\boldsymbol{\vartheta}), \quad (12)$$

$$\langle \mathcal{R}_E^2 \mathcal{R}_B \rangle(\boldsymbol{\vartheta}) \equiv \frac{1}{3} [\langle \mathcal{R}_\times \mathcal{R}_1 \mathcal{R}_1 \rangle + \langle \mathcal{R}_1 \mathcal{R}_\times \mathcal{R}_1 \rangle + \langle \mathcal{R}_1 \mathcal{R}_\times \mathcal{R}_1 \rangle](\boldsymbol{\vartheta}) = \frac{1}{4} \text{Im} [3 \langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R} \mathcal{R}^* \mathcal{R} \rangle + \langle \mathcal{R} \mathcal{R} \mathcal{R}^* \rangle](\boldsymbol{\vartheta}), \quad (13)$$

$$\langle \mathcal{R}_B^3 \rangle(\boldsymbol{\vartheta}) \equiv \langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_\times \rangle(\boldsymbol{\vartheta}) = \frac{1}{4} \text{Im} [-\langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R} \mathcal{R}^* \mathcal{R} \rangle + \langle \mathcal{R} \mathcal{R} \mathcal{R}^* \rangle](\boldsymbol{\vartheta}), \quad (14)$$

where equation (11) corresponds to pure E-mode correlations, and equation (14) to parity-violating third-order B-mode correlations. Equation (12) is a parity-invariant correlation between E and B modes, and equation (13) is a parity-violating correlation between E and B modes.

For brevity, the mixed terms (12) and (13) are generalized expressions which are sensitive to B modes in any of the annuli. Instead one can also consider more localized B-mode measures like

$$\langle \mathcal{R}_\times \mathcal{R}_\times \mathcal{R}_1 \rangle(\boldsymbol{\vartheta}) = \frac{1}{4} \text{Re} [\langle -\mathcal{R} \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle + \langle \mathcal{R} \mathcal{R}^* \mathcal{R} \rangle - \langle \mathcal{R} \mathcal{R} \mathcal{R}^* \rangle](\boldsymbol{\vartheta}), \quad (15)$$

which picks up correlations with B modes in the innermost and middle annulus, but is insensitive to B modes in the outer annulus. Such a localized test for B-mode correlations can help identifying the source of a B-mode contamination if equation (12) indicates the overall presence of B modes.

#### 4 THIRD-ORDER RING STATISTIC

In this section, we derive computationally advantageous expressions for the third-order ring statistic in terms of the shear 3PCF, and show their relation to the convergence bispectrum.

##### 4.1 Relation to the shear three-point functions

We rewrite the third-order ring statistic in terms of the shear 3PCF by starting from the definition in equation (7):

$$\begin{aligned} \langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) &= - \int_{\vartheta_1}^{\vartheta_2} d\theta_1 W(\theta_1; \vartheta_1, \vartheta_2) \int_{\vartheta_3}^{\vartheta_4} d\theta_2 W(\theta_2; \vartheta_3, \vartheta_4) \int_{\vartheta_5}^{\vartheta_6} d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) \\ &\quad \times \int_0^{2\pi} \frac{d\psi_3}{2\pi} \int_0^{2\pi} \frac{d\psi_2}{2\pi} \int_0^{2\pi} \frac{d\psi_1}{2\pi} e^{-2i(\psi_1 + \psi_2 + \psi_3)} \langle \gamma^c(\theta_1, \psi_1) \gamma^c(\theta_2, \psi_2) \gamma^c(\theta_3, \psi_3) \rangle. \end{aligned} \quad (16)$$

Noting that  $\mathbf{X}_j = \theta_j \exp(i\psi_j)$  and using equation (3), this can be rewritten as

$$\begin{aligned} \langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) &= \int \frac{d^2 X_1}{2\pi |\mathbf{X}_1|} W(|\mathbf{X}_1|; \vartheta_1, \vartheta_2) \int \frac{d^2 X_2}{2\pi |\mathbf{X}_2|} W(|\mathbf{X}_2|; \vartheta_3, \vartheta_4) \int \frac{d^2 X_3}{2\pi |\mathbf{X}_3|} W(|\mathbf{X}_3|; \vartheta_5, \vartheta_6) \\ &\quad \times \exp[2i(\alpha_1 + \alpha_2 + \alpha_3 - \psi_1 - \psi_2 - \psi_3)] \Gamma^{(0)}(\mathbf{X}_1 - \mathbf{X}_3, \mathbf{X}_2 - \mathbf{X}_3), \end{aligned} \quad (17)$$

where  $\Gamma^{(0)}$  is the shear 3PCF measured relative to the centroid, so that the  $\alpha_i$  are the directions of the point  $\mathbf{X}_i$  to the centroid  $\bar{\mathbf{X}} = (\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3)/3$ . Owing to circular symmetry, we can set  $\psi_3 = 0$ ; equivalently, one can use relative polar angles  $\Delta\psi_j = \psi_j - \psi_3$  and show that the integrand depends only on these relative angles.

As  $\Gamma$  is measured within discrete angular bins, while the weight functions and geometric factors in equation (17) can be evaluated continuously, it is numerically more stable to rewrite the third-order ring statistic such that only the three outermost integrals contain the shear 3PCF and the inner integrals can be evaluated numerically to arbitrary precision. With  $\mathbf{x}_j = \mathbf{X}_j - \mathbf{X}_3 = \theta_j e^{i\psi_j} - \theta_3$  for  $j = 1, 2$ ,

$$\begin{aligned} \langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) &= \frac{1}{(2\pi)^2} \int d^2 x_1 \int d^2 x_2 \Gamma^{(0)}(\mathbf{x}_1, \mathbf{x}_2) \int d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) W(|\mathbf{x}_1 + \boldsymbol{\theta}_3|; \vartheta_1, \vartheta_2) W(|\mathbf{x}_2 + \boldsymbol{\theta}_3|; \vartheta_3, \vartheta_4) \\ &\quad \times \frac{1}{|\mathbf{x}_1 + \boldsymbol{\theta}_3| |\mathbf{x}_2 + \boldsymbol{\theta}_3|} \exp[2i(\alpha_1 + \alpha_2 + \alpha_3 - \psi_1 - \psi_2)], \end{aligned} \quad (18)$$

where we have used  $\boldsymbol{\theta}_3$  to denote a complex number with zero imaginary part for consistency. We have  $e^{2i\alpha_j} = \mathbf{q}_j / \mathbf{q}_j^*$ , with  $\mathbf{q}_1 = (2\mathbf{x}_1 - \mathbf{x}_2)/3$ ,  $\mathbf{q}_2 = (2\mathbf{x}_2 - \mathbf{x}_1)/3$ ,  $\mathbf{q}_3 = -(\mathbf{x}_1 + \mathbf{x}_2)/3$  and  $e^{i\psi_j} = (\mathbf{x}_j + \boldsymbol{\theta}_3)/\theta_j$ , so that

$$e^{-2i\psi_j} = \frac{\mathbf{x}_j^* + \boldsymbol{\theta}_3^*}{\mathbf{x}_j + \boldsymbol{\theta}_3}. \quad (19)$$

Thus,

$$\begin{aligned} \langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) &= \frac{1}{(2\pi)^2} \int d^2 x_1 \int d^2 x_2 \Gamma^{(0)}(\mathbf{x}_1, \mathbf{x}_2) \int d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) W(|\mathbf{x}_1 + \boldsymbol{\theta}_3|; \vartheta_1, \vartheta_2) W(|\mathbf{x}_2 + \boldsymbol{\theta}_3|; \vartheta_3, \vartheta_4) \\ &\quad \times \frac{1}{|\mathbf{x}_1 + \boldsymbol{\theta}_3| |\mathbf{x}_2 + \boldsymbol{\theta}_3|} \frac{\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3}{\mathbf{q}_1^* \mathbf{q}_2^* \mathbf{q}_3^*} \frac{\mathbf{x}_1^* + \boldsymbol{\theta}_3^*}{\mathbf{x}_1 + \boldsymbol{\theta}_3} \frac{\mathbf{x}_2^* + \boldsymbol{\theta}_3^*}{\mathbf{x}_2 + \boldsymbol{\theta}_3}. \end{aligned} \quad (20)$$

Finally, if  $\varphi_i$  is the polar angle of  $\mathbf{x}_i$ , and  $\phi = \varphi_2 - \varphi_1$  is the angle between  $\mathbf{x}_2$  and  $\mathbf{x}_1$ , we obtain

$$\begin{aligned} \langle \mathcal{R}\mathcal{R}\mathcal{R} \rangle(\boldsymbol{\vartheta}) &= \frac{1}{(2\pi)^2} \int d\mathbf{x}_1 \int d\mathbf{x}_2 \int d\phi \Gamma^{(0)}(\mathbf{x}_1, \mathbf{x}_2, \phi) \\ &\times \int d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) \int d\varphi_1 W(|x_1 e^{i\varphi_1} + \theta_3|; \vartheta_1, \vartheta_2) W(|x_2 e^{i(\varphi_1+\phi)} + \theta_3|; \vartheta_3, \vartheta_4) \\ &\frac{1}{|\mathbf{x}_1 + \boldsymbol{\theta}_3||\mathbf{x}_2 + \boldsymbol{\theta}_3|} \frac{\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3 \mathbf{x}_1^* + \theta_3 \mathbf{x}_2^* + \theta_3}{\mathbf{q}_1^* \mathbf{q}_2^* \mathbf{q}_3^* \mathbf{x}_1 + \theta_3 \mathbf{x}_2 + \theta_3} \\ &\equiv \frac{1}{(2\pi)^2} \int d\mathbf{x}_1 \int d\mathbf{x}_2 \int d\phi \Gamma^{(0)}(\mathbf{x}_1, \mathbf{x}_2, \phi) \mathbf{Z}_0(\mathbf{x}_1, \mathbf{x}_2, \phi, \boldsymbol{\vartheta}), \end{aligned} \quad (21)$$

where we have defined the complex filter function  $\mathbf{Z}_0$  of the ring statistic in the last step. Note that the ratio of the  $\mathbf{q}$  values does not depend on  $\theta_3$  and thus the evaluation of the filter function can be further simplified by reversing the order of integration and moving this phase factor to the outer ( $\varphi_1$ ) integral.

Expressions for the other correlations required for E/B-mode separation, which contain a complex conjugate ring statistic  $\mathcal{R}^*$ , are derived analogously. For the correlation involving the complex conjugate shear at vertex  $\mathbf{X}_j$ , the resulting expression analogous to equation (21) contains  $\Gamma^{(j)}$  instead of  $\Gamma^{(0)}$ ,  $\mathbf{q}_j/\mathbf{q}_j^*$  is replaced by its complex conjugate (corresponding to  $\alpha_j \rightarrow -\alpha_j$  in equation 3), and for  $j = 1, 2$  the factor  $\mathbf{x}_j/\mathbf{x}_j^*$  is also replaced by its complex conjugate (corresponding to  $\psi_j \rightarrow -\psi_j$  in the equivalent of equation 16), e.g.

$$\langle \mathcal{R}^* \mathcal{R} \mathcal{R} \rangle(\boldsymbol{\vartheta}) = \frac{1}{(2\pi)^2} \int d\mathbf{x}_1 \int d\mathbf{x}_2 \int d\phi \Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \phi) \mathbf{Z}_1(\mathbf{x}_1, \mathbf{x}_2, \phi, \boldsymbol{\vartheta}), \quad (22)$$

with

$$\begin{aligned} \mathbf{Z}_1(\mathbf{x}_1, \mathbf{x}_2, \phi, \boldsymbol{\vartheta}) &= \int d\theta_3 W(\theta_3; \vartheta_5, \vartheta_6) \int d\varphi_1 W(|x_1 e^{i\varphi_1} + \theta_3|; \vartheta_1, \vartheta_2) W(|x_2 e^{i(\varphi_1+\phi)} + \theta_3|; \vartheta_3, \vartheta_4) \\ &\frac{1}{|\mathbf{x}_1 + \boldsymbol{\theta}_3||\mathbf{x}_2 + \boldsymbol{\theta}_3|} \frac{\mathbf{q}_1^* \mathbf{q}_2 \mathbf{q}_3 \mathbf{x}_1 + \theta_3 \mathbf{x}_2^* + \theta_3}{\mathbf{q}_1 \mathbf{q}_2^* \mathbf{q}_3^* \mathbf{x}_1^* + \theta_3 \mathbf{x}_2 + \theta_3}. \end{aligned} \quad (23)$$

In the computation of the ring statistic, one can choose any (normalized) radial weight function  $W$  that fulfils  $W(0; \vartheta_1, \vartheta_2) = 0$  even if  $\vartheta_1 = 0$  (as the separation in tangential/cross shear is ill defined on a circle of radius  $\theta = 0$ ). To be specific, we choose

$$W(\theta; \vartheta_i, \vartheta_j) = \frac{30(\theta - \vartheta_i)^2(\vartheta_j - \theta)^2}{(\vartheta_j - \vartheta_i)^5}, \quad (24)$$

as in the computation of the second-order ring statistic (Schneider & Kilbinger 2007). The shape of the third-order ring statistic filter functions  $\mathbf{Z}_{0,1}$  based on this choice for  $W$  is illustrated in Fig. 2.

## 4.2 Relation to the bispectrum

In order to rewrite the third-order ring statistic in terms of the bispectrum we first relate it to the lensing convergence field  $\kappa$ , which is easier to express in terms of the convergence bispectrum than the shear 3PCF (see Schneider et al. 2005, for details) as it contains fewer oscillatory phase factors. Expressing the ring statistic in terms of the convergence field also speeds up the measurement of  $\langle \mathcal{R}_E^3 \rangle$  in simulations considerably, as described below.

Consider the convergence field smoothed with a radially symmetric filter  $U_\vartheta(\theta)$  with characteristic scale  $\vartheta$ . If  $U_\vartheta(\theta)$  is a compensated filter  $\int d\theta \theta U_\vartheta(\theta) = 0$ , this convolution can be expressed in terms of the shear field as

$$\int d^2\theta' U_\vartheta(|\boldsymbol{\theta}'|) \kappa(\boldsymbol{\theta}') = \int d^2\theta Q_\vartheta(|\boldsymbol{\theta}'|) \gamma_t(\boldsymbol{\theta}'), \quad (25)$$

where  $U$  and  $Q$  are related by (Kaiser, Squires & Broadhurst 1995; Schneider 1996; Schneider & Kilbinger 2007)

$$Q_\vartheta(\theta) = \frac{2}{\theta^2} \int_0^\theta d\theta' U_\vartheta(\theta') - U_\vartheta(\theta) \quad \text{and} \quad U_\vartheta(\theta) = \int_\theta^\infty \frac{2d\theta'}{\theta'} Q_\vartheta(\theta') - Q_\vartheta(\theta). \quad (26)$$

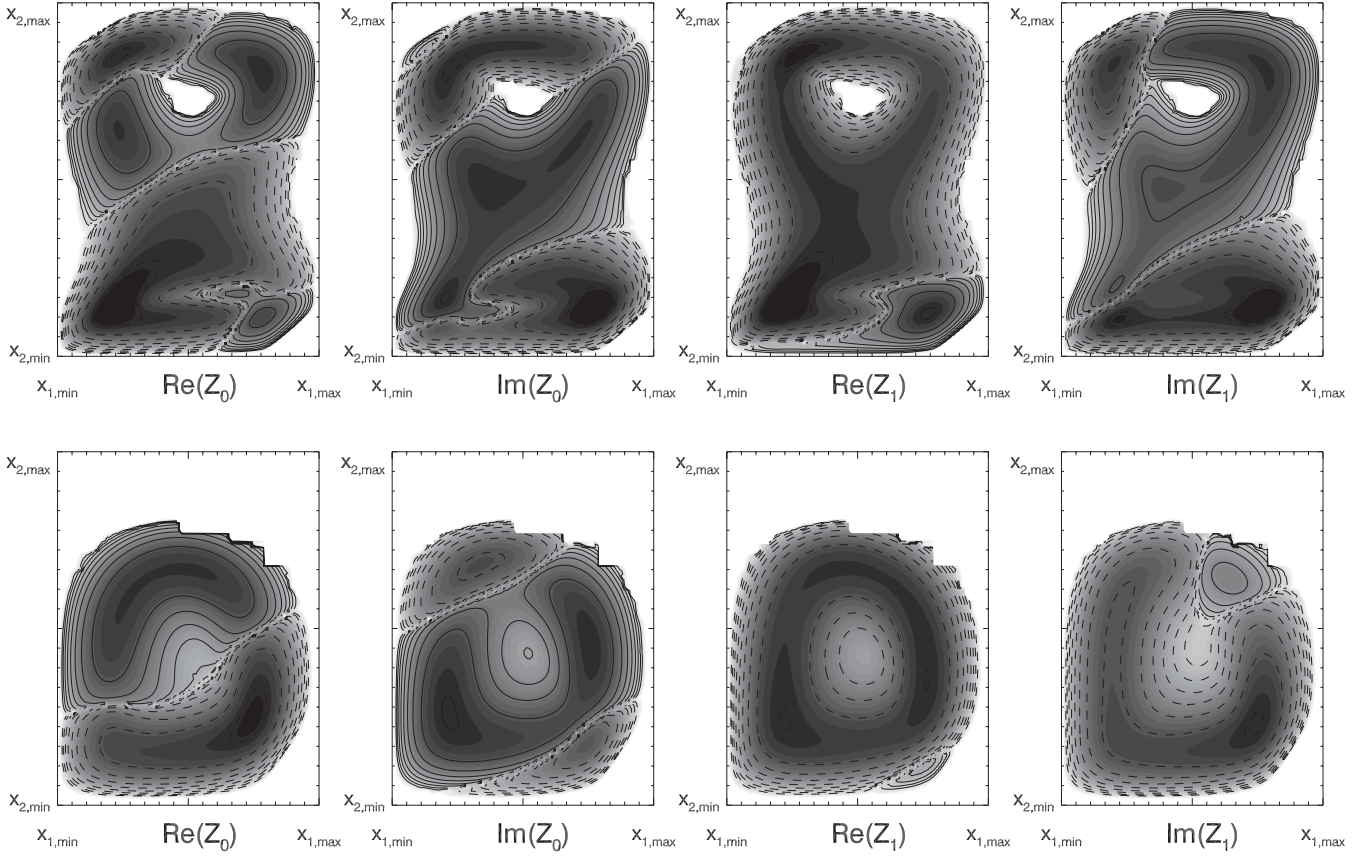
As shown in Schneider & Kilbinger (2007), the definition of the ring statistic  $\mathcal{R}$  (equation 5) is equivalent to an aperture mass  $M_{\text{ap}}(\vartheta_i, \vartheta_j)$  with two characteristic scales if

$$Q_{\vartheta_i, \vartheta_j}(\theta) = \frac{W(\theta; \vartheta_i, \vartheta_j)}{2\pi\theta}. \quad (27)$$

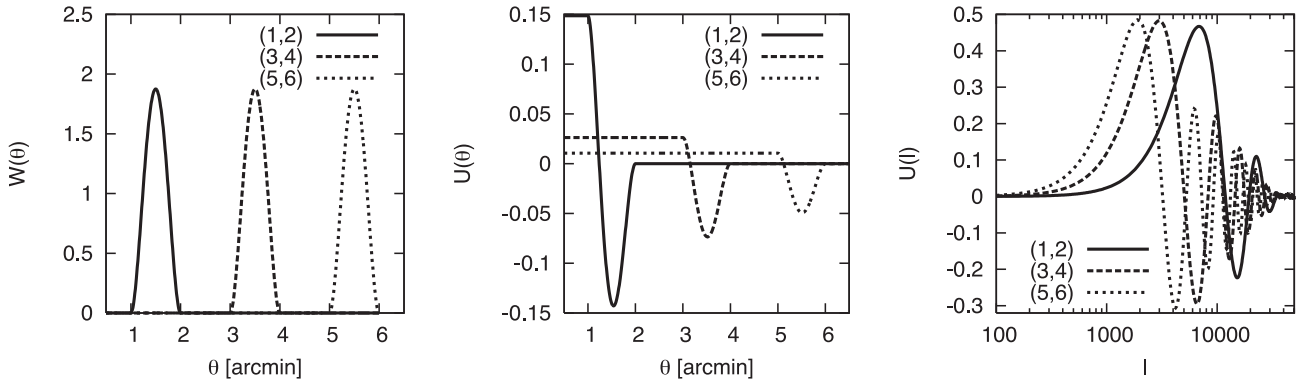
As the relation between the filter  $Q$  and  $U$  does not depend on the shape of  $Q$ , we can calculate the corresponding compensated filter  $U_{\vartheta_i, \vartheta_j}(\theta)$  as in equation (26). The left-hand and middle panels of Fig. 3 show the ring statistic filter  $W(\theta; \vartheta_i, \vartheta_j)$  and the corresponding aperture mass filter  $U_{\vartheta_i, \vartheta_j}(\theta)$  for different choices of ring radii  $(\vartheta_i, \vartheta_j)$ . As expected from equation (26),  $U$  is constant for  $\theta < \vartheta_i$ , then becomes negative and is zero for  $\theta > \vartheta_j$ .

Based on equations (25), (26) and (27), the third-order ring statistic of a pure E-mode field can be computed directly from simulated convergence maps by convolving the convergence field with different filters  $U_{\vartheta_i, \vartheta_j}$  and correlating three filtered maps. With this approach one does not need to calculate the shear 3PCF, which is computationally expensive (e.g. Jarvis et al. 2004). We stress that calculating the ring statistics from the convergence field is only possible for obtaining predictions from simulations, which are B-mode free by construction and allow for the direct measurement of the convergence.





**Figure 2.** Contours of the filter functions  $Z_{0,1}$  as a function of angular scales  $x_1 \in [\vartheta_5 - \vartheta_2, \vartheta_6 + \vartheta_2]$  and  $x_2 \in [\vartheta_5 - \vartheta_4, \vartheta_6 + \vartheta_4]$  for  $\vartheta = (1, 2, 3, 4, 5, 6)$  arcmin with  $\phi = \pi/8$  (top row) or  $\phi = \pi/4$  (bottom row). Contour lines are evenly spaced with separation 0.5 dex ranging from  $|Z| = 10^{-5}$  to  $10^{-1.5}$ , dashed lines indicate regions where  $Z$  is negative. The filter functions vanish if the triangle configuration  $(x_1, x_2, \phi)$  is not allowed in the ring statistic geometry (cf. Fig. 1).



**Figure 3.** Filter functions associated with  $\mathcal{R}$  for the same choice of ring radii  $(\vartheta_i, \vartheta_j)$  as in Fig. 2. The left-hand panel shows the radial filter function of the ring statistic,  $W(\theta; \vartheta_i, \vartheta_j)$ ; the corresponding compensated aperture mass filter function  $U_{\vartheta_i, \vartheta_j}(\theta)$  is shown in the middle panel, and the right-hand panel illustrates the corresponding Fourier transform  $\tilde{U}_{\vartheta_i, \vartheta_j}(l)$ .

Expressing  $\mathcal{R}_E$  as the convolution of  $\kappa$  and  $U_{\vartheta_i, \vartheta_j}$  also enables us to write down the third-order ring statistic of a pure E-mode field in terms of the convergence bispectrum  $B_\kappa(l_1, l_2, l_3)$  (cf. Schneider et al. 2005):

$$\langle \mathcal{R}_E^3 \rangle(\vartheta) = \frac{1}{(2\pi)^3} \int dl_1 l_1 \int dl_2 l_2 \int d\phi B_\kappa \left( l_1, l_2, \sqrt{l_1^2 + l_2^2 - 2l_1 l_2 \cos \phi} \right) \tilde{U}_{\vartheta_1, \vartheta_2}(l_1) \tilde{U}_{\vartheta_3, \vartheta_4}(l_2) \tilde{U}_{\vartheta_5, \vartheta_6} \left( \sqrt{l_1^2 + l_2^2 - 2l_1 l_2 \cos \phi} \right), \quad (28)$$

with the Fourier-transformed filter function  $\tilde{U}(l) = \int d\theta \theta J_0(l\theta) U(\theta)$ . The bispectrum filter functions for the third-order ring statistic are illustrated in the right-hand panel of Fig. 3.

## 5 CONCLUSION

Upcoming lensing surveys will provide data of unprecedented quality and enable us to conduct robust measurements of cosmic shear beyond the two-point level. These higher order statistics contribute substantial information to cosmological constraints by breaking parameter degeneracies when combined with second-order shear statistics. Furthermore, three-point statistics have the potential to improve our understanding of systematic effects in the data, e.g. the detection of third-order B modes can be an additional indicator for unsolved problems in the data analysis.

When extracting third-order information from a high-quality data set, it is therefore essential to use robust and unbiased theoretical methods that meet the quality of the data. We have introduced the third-order ring statistic, which separates the shear 3PCF into third-order E-/B-mode correlations on a finite interval  $[\theta_{\min}, \theta_{\max}]$ . Hence this statistic does not require knowledge of the 3PCF down to zero lag, where it is impossible to measure. Thus, unlike the third-order aperture mass statistics, it is not affected by apparent E-/B-mode mixing (Kilbinger et al. 2006).

Our main results are equations (21), (22) and (23), which give compact expressions for the third-order ring statistic in terms of the shear 3PCF. Furthermore, in Section 4.2 we give convenient expressions for computing the E-mode ring statistic from numerical simulations, and from the convergence bispectrum which facilitate the comparison with theoretical models for weak lensing three-point statistics (e.g. Valageas, Sato & Nishimichi 2011).

In addition to the cosmological information contained in the E-mode signal, our expression for third-order B-mode correlations opens a new window to detect remaining systematics in the data. For example, the various permutations of  $\langle \mathcal{R}_E \mathcal{R}_B \mathcal{R}_E \rangle$  allow for an association of B modes with specific angular scales.

For the analysis of future shear 3PCF measurements, we recommend using equation (11) to obtain a clean third-order E-mode signal, and equations (12) and (15) to test for remaining B-mode correlations.

## ACKNOWLEDGMENTS

We thank Bhuvnesh Jain, Chris Hirata and Mike Jarvis for useful discussions.

This work was supported by the Deutsche Forschungsgemeinschaft under the Transregional Collaborative Research Center TR-33 ‘The Dark Universe’. EK is supported by the US National Science Foundation (AST-0807337), the US Department of Energy (DE-FG03-02-ER40701) and the David and Lucile Packard Foundation.

## REFERENCES

- Bunn E. F., Zaldarriaga M., Tegmark M., de Oliveira-Costa A., 2003, *Phys. Rev. D*, 67, 023501
- Crittenden R. G., Natarajan P., Pen U., Theuns T., 2002, *ApJ*, 568, 20
- Eifler T., Schneider P., Krause E., 2010, *A&A*, 510, A7
- Fu L. et al., 2008, *A&A*, 479, 9
- Huff E. M., Eifler T., Hirata C. M., Mandelbaum R., Schlegel D., Seljak U., 2011, preprint (arXiv:1112.3143)
- Ivezic Z. et al., 2008, preprint (arXiv:0805.2366)
- Jarvis M., Bernstein G., Jain B., 2004, *MNRAS*, 352, 338
- Kaiser N., Squires G., Broadhurst T., 1995, *ApJ*, 449, 460
- Kilbinger M., Schneider P., Eifler T., 2006, *A&A*, 457, 15
- Laureijs R. et al., 2011, preprint (arXiv:1110.3193)
- Schneider P., 1996, *MNRAS*, 283, 837
- Schneider P., Kilbinger M., 2007, *A&A*, 462, 841
- Schneider P., Lombardi M., 2003, *A&A*, 397, 809
- Schneider P., van Waerbeke L., Jain B., Kruse G., 1998, *MNRAS*, 296, 873
- Schneider P., van Waerbeke L., Mellier Y., 2002, *A&A*, 389, 729
- Schneider P., Kilbinger M., Lombardi M., 2005, *A&A*, 431, 9
- Schneider P., Eifler T., Krause E., 2010, *A&A*, 520, A116
- Schrabback T. et al., 2010, *A&A*, 516, A63
- Shi X., Schneider P., Joachimi B., 2011, *A&A*, 533, A48
- Takada M., Jain B., 2003, *ApJ*, 583, L49
- Takada M., Jain B., 2004, *MNRAS*, 348, 897
- Valageas P., Sato M., Nishimichi T., 2012, *A&A*, 541, 161
- Zaldarriaga M., Scoccimarro R., 2003, *ApJ*, 584, 559

This paper has been typeset from a  $\text{\LaTeX}$  file prepared by the author.